An Interactive Algorithm to Deal with Inconsistencies in the Representation of Cardinal Information

Brice Mayag[1], Michel Grabisch[1], and Christophe Labreuche[2]

[1] University of Paris 1, 106-112 Boulevard de l'Hôpital, 75013 Paris, France
bmayag@yahoo.fr, michel.grabisch@univ-paris1.fr
[2] T.R.T France, 1 avenue Augustin Fresnel, 91767 Palaiseau Cedex France
christophe.labreuche@thalesgroup.com

Abstract. We present a new interactive algorithm allowing to solve the inconsistencies problem, when the preferences of a decision maker cannot be representable by a numerical function. This algorithm is based on technics of linear programming and the type of preferences we use are cardinal information.

Keywords: Decision making, Preference modeling, Cardinal information, Inconsistencies.

1 Introduction

Decision making aims at helping a decision maker (DM) to select one or more alternatives among several alternatives. During this process, and in many situations, it is important for the DM to construct a preference relation over the set of all alternatives X. Many models have been developed to construct this preference. Some of them, like utility theory, look for a numerical function with good properties (arithmetic mean, Choquet integral, belief functions, ...) which is able to represent faithfully the preferences of the DM on X. This representation requires sometimes to ask to the DM an initial preference on X or when X is very large, a preferential information on a reference subset $X' \subseteq X$.

In this paper, we ask the DM to give, using pairwise comparisons, a cardinal information (a preferential information given with preference intensity) on X and then we test if this preferential information is consistent with a numerical function. If the test leads to inconsistencies, how to help the DM to modify his preferences in order to represent his cardinal information? To answer this question it is desirable to have recommendations understandable by any DM. This is not true with the different theorems [10,11,12,7] on the representation of cardinal information by a numerical function. Indeed, these characterizations are based on the notion of cyclones [5,13] (specific cycles) which is very complex to understand and to detect. Therefore, an alternative to these theorems is to use methods of dealing with inconsistencies based on technics of linear programming [2,8,9].

E. Hüllermeier, R. Kruse, and F. Hoffmann (Eds.): IPMU 2010, Part I, CCIS 80, pp. 148–157, 2010.

We propose a new interactive algorithm for inconsistency management with cardinal information. Our approach is not in the spirit of the determination of an irreducible inconsistent system (ISS) [3], but to use simple and intuitive methods of constraints relaxation when a linear program is infeasible. The recommendations we suggest to the DM are based on the concepts of augmentation and reduction of a preference able to causing an inconsistency.

The paper is organized as follows: the next section introduces the basic notions we need, then we present in Section 3 our algorithm and we end by an illustrative example.

2 Representation of a Cardinal Information

Let X be a finite set of alternatives (or actions, options). We assume that, given two alternatives x and y the DM is able to judge the difference of attractiveness between x and y when he strictly prefers x to y. Like in the MACBETH [1,4] and GRIP [6] methodologies in Muticriteria Decision Analysis, the difference of attractiveness will be provided under the form of semantic categories d_s, $s = 1, \dots, q$ defined so that, if $s < t$, any difference of attractiveness in the class d_s is smaller than any difference of attractiveness in the class d_t. MACBETH approach uses the following six semantic categories: d_1 = very weak, d_2 = weak, d_3 = moderate, d_4 = strong, d_5 = very strong, d_6 = extreme. If there is no ambiguity, a category d_s will be simply designated by s.

Under these hypotheses, the preferences given by the DM is expressed by the following relations:

- $P = \{(x, y) \in X \times X$: the DM strictly prefers x to $y\}$, P is an asymmetric relation;
- $I = \{(x, y) \in X \times X$: the DM is indifferent between x and $y\}$, I is an reflexive and symmetric relation;
- For the semantic categories "d_s", "d_t", $s, t \in \{1, ..., q\}$, $s \leq t$,
  $P_{st} = \{(x, y) \in P$ such that the DM judges the difference of attractiveness between x and y as belonging from the class "d_s" to the class "d_t" $\}$. When $s < t$, P_{st} expresses some hesitation.

Remark 1. In this paper, the relation $P \cup I$ is not necessarily complete.

Definition 1. *The cardinal information on X is the structure $\{P, I, \{P_{st}\}_{s \leq t}\}$.*

We will suppose P to be nonempty for any cardinal information $\{P, I, \{P_{st}\}_{s \leq t}\}$ ("non triviality axiom") and $P = \bigcup_{s,t} P_{st}$. Remark that if the DM wants to say that x is strictly preferred to y, but he hesitates completely on the category, then he will write $xP_{1q}y$.

A cardinal information$\{P, I, \{P_{st}\}_{s \leq t}\}$ is said to be *representable by a numerical function $f : X \to \mathbb{R}_+$* if the following conditions are satisfied: $\forall x, y, z, w \in X$,

$\forall s, t, u, v \in \{1, \dots, q\}$ such that $u \leq v < s \leq t$,

$$x \; I \; y \Rightarrow f(x) = f(y), \tag{1}$$

$$x \; P \; y \Rightarrow f(x) > f(y), \tag{2}$$

$$\left.\begin{matrix}(x, y) \in P_{st} \\ (z, w) \in P_{uv}\end{matrix}\right\} \Rightarrow f(x) - f(y) > f(z) - f(w) \tag{3}$$

De Corte proved in [1] that the previous conditions are equivalent to the existence of q thresholds $\sigma_1, \dots, \sigma_q$ such that:

$$\forall (x, y) \in I : f(x) = f(y), \tag{4}$$

$$\forall s, t \in \{1, \dots, q\}, \; s \leq t, \; \forall (x, y) \in P_{st} : \sigma_s < f(x) - f(y), \tag{5}$$

$$\forall s, t \in \{1, \dots, q-1\}, \; s \leq t, \; \forall (x, y) \in P_{st} : f(x) - f(y) < \sigma_{t+1}, \tag{6}$$

$$0 < \sigma_1 < \sigma_2 < \dots < \sigma_q \tag{7}$$

Note that in this representation, the relation (2) disappears so that relation P is no more used explicitly. To know if a cardinal information $\{P, I, \{P_{st}\}_{s \leq t}\}$ on X is representable by a function f, we use the following linear program PL_1:

$$\text{PL}_1 \begin{cases} \text{Min} & f(x_0) \\ s.t. & f(x) = f(y), & \forall (x, y) \in I & (c1) \\ & \sigma_i + d_{\min} \leq f(x) - f(y), & \forall (x, y) \in P_{ij}, \forall i, j \in \{1, \dots, q\}, \; i \leq j & (c2) \\ & f(x) - f(y) \leq \sigma_{j+1} - d_{\min}, & \forall (x, y) \in P_{ij}, \forall i, j \in \{1, \dots, q-1\}, \; i \leq j & (c3) \\ & d_{\min} \leq \sigma_1 & & (c4) \\ & \sigma_{i-1} + d_{\min} \leq \sigma_i, & \forall i \in \{2, \dots, q\} & (c5) \end{cases}$$

where x_0 is an alternative of X arbitrarily chosen, and $d_{\min}$ an arbitrary strictly positive constant.

Now, when the cardinal information is inconsistent, i.e. the program PL_1 is infeasible, how to elaborate recommendations for the DM in order to have the consistent judgements? A natural solution is to provide these recommendations by using characterization theorems of the representation of a cardinal information studied in [7,11,12]. But, all these theorems are based on the more complex and specific cycle called "cyclone" [5], which would be difficult to grasp for a DM. Our aim is to propose a new interactive method able to generate recommendations for the DM when PL_1 is infeasible.

3 Our Algorithm

3.1 Step 1: Find the Minimal Number of Constraints to Be Relaxed

To make PL_1 feasible, we choose to relax some of its constraints which can cause an inconsistency. To do this, we associate to each constraint l of PL_1, a binary variable β_l allowing to know whether if the constraint l has to be relaxed or not. The options x and y in constraint l are denoted by x_l and x'_l. Then we find the minimal number of constraints which we will relax by solving the following linear program PL_2:

$$
\mathrm{PL}_2 \begin{cases}
\mathrm{Min} \sum_{l \in \mathbb{N}_{1,c}} \beta_l & & \\
s.t.\ \ f(x_l) - f(x'_l) + M\beta_l \geq 0, & \forall (x_l, x'_l) \in I,\ l \in \mathbb{N}_{1,r^+} & (c1')_1 \\
f(x_l) - f(x'_l) - M\beta_l \leq 0, & \forall (x_l, x'_l) \in I,\ l \in \mathbb{N}_{(r^+ +1),r} & (c1')_2 \\
\sigma_i + d_{\min} \leq f(x_l) - f(x'_l) + M\beta_l, & \forall (x_l, x'_l) \in P_{ij}, \forall i,j \in \mathbb{N}_{1,q}, l \in \mathbb{N}_{(r+1),(r+p_1)} & (c2') \\
f(x_l) - f(x'_l) - M\beta_l \leq \sigma_{j+1} - d_{\min}, & \forall (x_l, x'_l) \in P_{ij}, \forall i,j \in \mathbb{N}_{1,q-1}, l \in \mathbb{N}_{r+p_1+1,c} & (c3') \\
d_{\min} \leq \sigma_1 & & (c4) \\
\sigma_{i-1} + d_{\min} \leq \sigma_i, & \forall i \in \mathbb{N}_{2,q} & (c5) \\
\beta_l \in \{0,1\}, & \forall l \in \mathbb{N}_{1,c} & (c6)
\end{cases}
$$

where

- each constraint $f(x) - f(y) = 0$ of PL_1 is replaced in PL_2 by the following two constraints:
  (i) $f(x_l) - f(x'_l) + M\beta_l \geq 0,\ \ 1 \leq l \leq r^+$ (type $(c1')_1$);
  (ii) $f(x_{l'}) - f(x'_{l'}) - M\beta_{l'} \leq 0,\ \ r^+ \leq l' \leq r$ (type $(c1')_2$);
  such that $x_l = x_{l'}$ and $x'_l = x'_{l'}$. It is obvious that these two inequalities are always satisfied when $\beta_l = \beta_{l'} = 1$.
- M is a positive large number.
- $r = r^+ + r^-$ with respectively r^+ and r^- the number of constraints of $(c1')_1$ and $(c1')_2$. $r^+ = r^-$ is the number of constraints of type $(c1)$.
- p_1: the number of constraints of type $(c2)'$ corresponding to the number of constraints of $(c2)$ in PL_1.
- p_2: the number of constraints of $(c3)'$ corresponding to the number of constraints of $(c3)$ in PL_1.
- $c = r + p_1 + p_2$;
- $\forall s, t \in \mathbb{N},\ s \leq t,\ \mathbb{N}_{s,t} = \{s, s+1, \ldots, t\}$.

3.2 Step 2: Relaxation by Augmentation or Reduction by *p* Categories

In this section, we show how to relax each constraint which has its binary variable β_l equals to 1. We suggest two types of relaxation: an increase or decrease of categories and we justify this by the following:

1. Suppose that a preference $(x, y) \in P_{ij}$ causes an inconsistency in PL_1. If the modification of this judgement can restore the consistency, it seems natural to ask the DM to adopt one of these two recommendations:
   - If $(x, y) \in P_{ij}$ belongs to the set of constraints of $(c3')$, increase the category j by replacing this preference by $(x, y) \in P_{ij'}$ with $j < j'$;
   - If $(x, y) \in P_{ij}$ belongs to the set of constraints of $(c2')$, decrease the category i by replacing this preference by $(x, y) \in P_{i'j}$ with $i' < i$.
2. If an indifférence (x_l, x'_l) causes an inconsistency, then PL_2 satisfies either $f(x_l) - f(x'_l) < 0$ or $f(x_l) - f(x'_l) > 0$ (corresponding to $\beta_l = 1$). Therefore if the inequality $f(x_l) - f(x'_l) > 0$ is satisfied in PL_2, we recommend to the DM to change $(x_l, x'_l) \in I$ by $(x_l, x'_l) \in P_{1p}$ where p will be a category to be determined.

We need the following notation in the formal Definition 2 of relaxation by augmentation or reduction by p categories:

- the judgement "$(x, y) \in P_{ij}$" will be represented by the element (x, y, i, j) of $X \times X \times \mathbb{N}_{1,q} \times \mathbb{N}_{1,q}$;
- the judgement "$(x, y) \in I$" will be represented by the element $(x, y, 0, 0)$ of $X \times X \times \mathbb{N} \times \mathbb{N}$.

Definition 2.

1. "A reduction of the judgement (x, y, i, j) with p categories" *is the substitution of this judgement by:*
   (a) the judgement $(x, y, i - p, j)$ *if* $(1 \leq p < i)$*;*
   (b) the judgement $(y, x, 1, p)$ *if* $i = j = 0$.
2. "An augmentation of the judgement (x, y, i, j) of p categories" $(1 \leq p \leq q - j)$ *is the substitution of this judgement by:*
   (a) the judgement $(x, y, 1, p)$ *if* $i = j = 0$*;*
   (b) the judgement $(x, y, i, j + p)$ *otherwise.*

Using the previous notions, we distinguish two cases:

(i) The judgement is $(x_l, x'_l) \in I$:
  - If the binary variable $\beta_l = 1$ of PL_2 is associated to the constraint $f(x_l) - f(x'_l) + M\beta_l \geq 0$ derived from the preference $(x_l, x'_l) \in I$, then the corresponding relaxation is a reduction of the judgement by p categories. We denote by C_{1-1} the set of all "l" satisfying these conditions.
  - On the other side, if $\beta_l = 1$ is associated to the constraint $f(x_l) - f(x'_l) - M\beta_l \leq 0$ with $(x_l, x'_l) \in I$, then the proposition of relaxation will be an augmentation of the judgement by p categories. We denote by C_{1-2} the set of all "l" satisfying this type of conditions.

(ii) The judgement is $(x_l, x'_l) \in P_{ij}$:
  - If the binary variable $\beta_l = 1$ of PL_2 is associated to the constraint $\sigma_i + d_{\min} \leq f(x_l) - f(x'_l) + M\beta_l$, then we apply the reduction of the judgement. Let us denote by C_2 the set of all "l" satisfying this type of conditions.
  - If $\beta_l = 1$ corresponds to $f(x_l) - f(x'_l) - M\beta_l \leq \sigma_{j+1} - d_{\min}$, then we recommend an augmentation of judgement. Let C_3 the set of all "l" satisfying these type of conditions.

3.3 Step 3: Determination of the Number of Categories p Used in the Relaxation

In this section, we suppose the set M of m constraints, which can cause an inconsistency, have been determined through the linear program PL_2. To know for each element l of M, the number of categories necessary for its relaxation by augmentation or reduction, we introduce the binary variables ε^l_k as follows:

1. If the modification of the preference $(x_l, x'_l) \in I$ requires an augmentation of categories, we replace in PL_1 the constraint $f(x_l) - f(x'_l) = 0$ associated to this judgement by the following constraints:

$$\begin{cases} f(x_l) - f(x'_l) \geq 0 \\ f(x_l) - f(x'_l) \leq \sigma_k - d_{\min} + M\ \varepsilon^l_k, \quad \forall k \in \mathbb{N}_{2,q} \end{cases} \tag{8}$$

   Let $h = \sum_{k \in \mathbb{N}_{2,q}} \varepsilon^l_k$.
   - If $h < q-1$, then we recommend to the DM an augmentation of $(x_l, x'_l) \in I$ with $(h+1)$ categories;
   - Otherwise, we suggest him to remove the judgement $(x_l, x'_l) \in I$ in the cardinal information.
2. If the judgement $(x_l, x'_l) \in P_{ij}$ requires an augmentation of categories, we replace in PL_1 the constraint $f(x_l) - f(x'_l) \leq \sigma_{j+1} - d_{\min}$ associated to this preference by the constraints

$$f(x_l) - f(x'_l) \leq \sigma_{j+k} - d_{\min} + M\ \varepsilon^l_k, \quad \forall k \in \mathbb{N}_{2,q-j} \tag{9}$$

   Let $h = \sum_{k \in \mathbb{N}_{2,q-j}} \varepsilon^l_k$.
   - If $h < q-j-1$, we recommend to the DM an augmentation of $(x_l, x'_l) \in P_{ij}$ by $(h+1)$ categories;
   - Otherwise, we propose him to remove the judgement $(x_l, x'_l) \in P_{ij}$.
3. If the preference $(x_l, x'_l) \in I$ requires a reduction of categories, we replace in PL_1 the corresponding constraint $f(x_l) - f(x'_l) = 0$ by the following constraint

$$\begin{cases} f(x'_l) - f(x_l) \geq 0 \\ f(x'_l) - f(x_l) \leq \sigma_k - d_{\min} + M\ \varepsilon^l_k, \quad \forall k \in \mathbb{N}_{2,q} \end{cases} \tag{10}$$

   Let $h = \sum_{k \in \mathbb{N}_{2,q}} \varepsilon^l_k$.
   - If $h < q-1$, we suggest to the DM a reduction of $(x_l, x'_l) \in I$ by $(h+1)$ categories;
   - Otherwise, we suggest to remove the preference $(x_l, x'_l) \in I$.
4. If the judgement $(x_l, x'_l) \in P_{ij}$ requires a reduction of categories, we replace in PL_1 the corresponding constraint $\sigma_i + d_{\min} \leq f(x_l) - f(x'_l)$ by the following:

$$\sigma_{i-k} + d_{\min} \leq f(x_l) - f(x'_l) + M\ \varepsilon^l_k, \quad \forall k \in \mathbb{N}_{1,i-1} \tag{11}$$

   Let $h = \sum_{k \in \mathbb{N}_{1,i-1}} \varepsilon^l_k$.
   - If $h < i-1$, we recommend to the DM a reduction of $(x_l, x'_l) \in P_{ij}$ with $(h+1)$ categories;
   - Otherwise, suggest him to remove $(x_l, x'_l) \in P_{ij}$.

The binary variables ε_k^l introduced in equations (8) to (11) are determined by the following linear program PL_3:

$$\mathrm{PL}_3 \begin{cases} \text{Min} \displaystyle\sum_{k\in\mathbb{N}_{2,q},l\in C_{1-1}} \varepsilon_k^l + \sum_{k\in\mathbb{N}_{1,q-j},l\in C_2} \varepsilon_k^l + \sum_{k\in\mathbb{N}_{2,q},l\in C_{1-2}} \varepsilon_k^l + \sum_{k\in\mathbb{N}_{1,i-1},l\in C_3} \varepsilon_k^l \\ \text{s.t. } \mathrm{S}_{\text{Relaxed constraints}} \\ \quad\quad \mathrm{S}_{\mathrm{PL}_1} \setminus \mathrm{S}_{\mathrm{PL}_1(m)} \\ \quad\quad \varepsilon_k^l \in \{0,1\}, \forall k \in \mathbb{N}_{2,q}, l \in C_{1-1} \\ \quad\quad \varepsilon_k^l \in \{0,1\}, \forall k \in \mathbb{N}_{1,q-j}, l \in C_2 \\ \quad\quad \varepsilon_k^l \in \{0,1\}, \forall k \in \mathbb{N}_{2,q}, l \in C_{1-2} \\ \quad\quad \varepsilon_k^l \in \{0,1\}, \forall k \in \mathbb{N}_{1,i-1}, l \in C_3 \end{cases}$$

where

- $\mathrm{S}_{\mathrm{PL}_1}$ represents all the constraints of PL_1.
- $\mathrm{S}_{\mathrm{PL}_1(m)}$ represents a subset of $\mathrm{S}_{\mathrm{PL}_1}$ formed by the constraints associated to the constraints of M build by PL_2 that cause an inconsistency.
- $\mathrm{S}_{\text{Relaxed constraints}}$ represents the system formed by all the constraints introduced in (8) to (11).

3.4 Step 4: The Interaction with the DM

We have seen in the previous sections that, if the cardinal information given by the DM is inconsistent, then the linear program PL_3 is solved and its solution is presented to the DM as recommendations to repair the inconsistencies. Therefore, for each judgement $(x_l, x'_l) \in (P \cup I)$ causing an inconsistency, we suggest him an augmentation or reduction of categories to make consistent judgements representable by a numerical function f. Let R be the set of recommendations (judgements with augmentation or reduction) proposed to the DM. In our algorithm, the DM can adopt one of these two positions:

1. *the DM does not agree with the recommendations proposed.*
   He builds a subset $R' \subseteq R$ of judgements for which he decides to conserve his initial judgement. For each element (x_l, x'_l) of R', the DM has no intention to relax the constraint corresponding to (x_l, x'_l) in PL_1. Therefore, the linear program PL_2 will be launched again by considering these constraints as satisfied constraints (by removing their binary variables β_l). There are two possibilities:
   (a) PL_2 with these constraints has a solution. Then we compute the new recommendations. We are thus either in situation 1 or 2.
   (b) PL_2 with these constraints has no solution. This means that the DM cannot conserve R' since they are inconsistent. He needs thus to change R'.
2. *the DM agrees with the recommendation proposed.*
   The linear program PL_1 is launched again by taking into account the new consistent cardinal information given by the DM.

The algorithm is represented by the Figure 1.

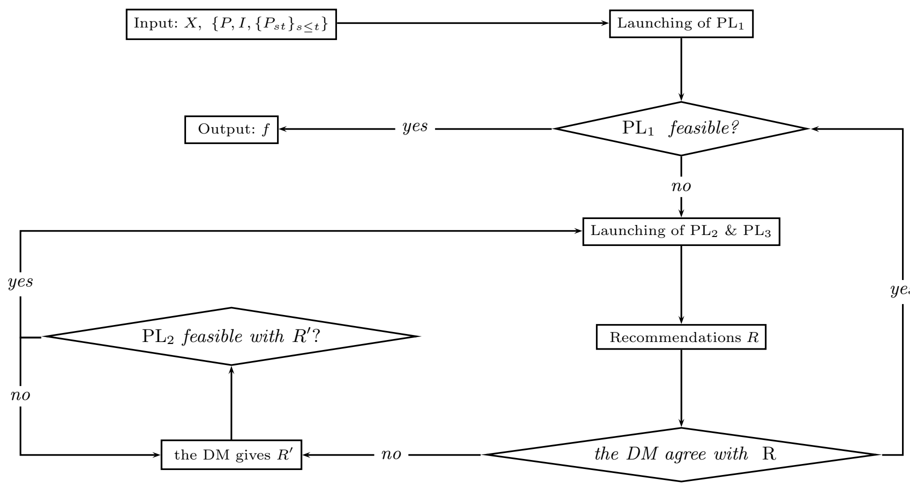

Fig. 1. Interactive algorithm of dealing with inconsistencies

4 An Illustrative Example

$X = \{x_1;\ x_2;\ x_3;\ x_4;\ x_5;\ x_6\}$; $q = 6$. Suppose that the DM gives the following preferences: $I = \{(x_2, x_3); (x_1, x_6)\}$; $P_3 = \{(x_5, x_6)\}$; $P_{12} = \{(x_1, x_3)\}$; $P_{24} = \{(x_1, x_5)\}$; $P_{46} = \{(x_3, x_5)\}$. The consistency of the cardinal information $\{I,\ P_3,\ P_{12},\ P_{24},\ P_{46}\}$ is tested through the linear program PL_1:

$$\mathrm{PL}_1 \begin{cases} \text{Min } f(x_1) \\ \text{s.t.} \\ f(x_2) - f(x_3) = 0 \\ f(x_1) - f(x_6) = 0 \\ \sigma_3 + d_{\min} \le f(x_5) - f(x_6) \\ \sigma_1 + d_{\min} \le f(x_1) - f(x_3) \\ \sigma_2 + d_{\min} \le f(x_1) - f(x_5) \\ \sigma_4 + d_{\min} \le f(x_3) - f(x_5) \\ f(x_5) - f(x_6) \le \sigma_4 - d_{\min} \\ f(x_1) - f(x_3) \le \sigma_3 - d_{\min} \\ f(x_1) - f(x_5) \le \sigma_5 - d_{\min} \\ d_{\min} \le \sigma_1, \quad \sigma_i + d_{\min} \le \sigma_{i+1}, i = 1, \ldots, 5 \end{cases}$$

For this test, we set $d_{\min} = 0.001$ and we get PL_1 infeasible. Therefore the algorithm launches the linear program PL_2 in order to find the minimal number of constraints to be relaxed:

$$\mathrm{PL}_2 \begin{cases} \text{Min } \sum_{l=1}^{11} \beta_l \\ \text{s.t.} \\ f(x_2) - f(x_3) + M\beta_1 \geq 0 \\ f(x_1) - f(x_6) + M\beta_2 \geq 0 \\ f(x_2) - f(x_3) - M\beta_3 \leq 0 \\ f(x_1) - f(x_6) - M\beta_4 \leq 0 \\ \sigma_3 + d_{\min} \leq f(x_5) - f(x_6) + M\beta_5 \\ \sigma_1 + d_{\min} \leq f(x_1) - f(x_3) + M\beta_6 \\ \sigma_2 + d_{\min} \leq f(x_1) - f(x_5) + M\beta_7 \\ \sigma_4 + d_{\min} \leq f(x_3) - f(x_5) + M\beta_8 \\ f(x_5) - f(x_6) - M\beta_9 \leq \sigma_4 - d_{\min} \\ f(x_1) - f(x_3) - M\beta_{10} \leq \sigma_3 - d_{\min} \\ f(x_1) - f(x_5) - M\beta_{11} \leq \sigma_5 - d_{\min} \\ d_{\min} \leq \sigma_1, \quad \sigma_i + d_{\min} \leq \sigma_{i+1}, i = 1, \ldots, 5 \\ \beta_l \in \{0, 1\}, \forall l \in \{1, \ldots, 11\} \end{cases}$$

The solution gives $\beta_4 = 1$ and $\beta_l = 0$ for $l \neq 4$. So the only constraint which need to be relaxed is $f(x_1) - f(x_6) = 0$ and its relation corresponds to an augmentation of p categories of the judgement $(x_1, x_6) \in I$. The number p is given by PL_3:

$$\mathrm{PL}_3 \begin{cases} \text{Min } \sum_{l=2}^{6} \varepsilon_l \\ \text{s.t.} \\ f(x_2) - f(x_3) = 0 \\ f(x_1) - f(x_6) \geq 0 \\ f(x_1) - f(x_6) \leq \sigma_2 - d_{\min} - M\varepsilon_2 \\ f(x_1) - f(x_6) \leq \sigma_3 - d_{\min} - M\varepsilon_3 \\ f(x_1) - f(x_6) \leq \sigma_4 - d_{\min} - M\varepsilon_4 \\ f(x_1) - f(x_6) \leq \sigma_5 - d_{\min} - M\varepsilon_5 \\ f(x_1) - f(x_6) \leq \sigma_6 - d_{\min} - M\varepsilon_6 \\ \sigma_3 + d_{\min} \leq f(x_5) - f(x_6) \\ \sigma_1 + d_{\min} \leq f(x_1) - f(x_3) \\ \sigma_2 + d_{\min} \leq f(x_1) - f(x_5) \\ \sigma_4 + d_{\min} \leq f(x_3) - f(x_5) \\ f(x_5) - f(x_6) \leq \sigma_4 - d_{\min} \\ f(x_1) - f(x_3) \leq \sigma_3 - d_{\min} \\ f(x_1) - f(x_5) \leq \sigma_5 - d_{\min} \\ d_{\min} \leq \sigma_1, \quad \sigma_i + d_{\min} \leq \sigma_{i+1}, i = 1, \ldots, 5 \\ \varepsilon_l \in \{0, 1\}, \forall l \in \{2, \ldots, 6\} \end{cases}$$

A solution of PL_3 gives $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_5 = \varepsilon_6 = 0$. Therefore we suggest to the DM an augmentation of $(x_1, x_6) \in I$ by 4 categories, i.e. replace $(x_1, x_6) \in I$ by $(x_1, x_6) \in P_{14}$. The DM accepts this recommendation and the new cardinal information $I = \{(x_2, x_3)\}$; $P_3 = \{(x_5, x_6)\}$; $P_{12} = \{(x_1, x_3)\}$; $P_{14} = \{(x_1, x_6)\}$; $P_{24} = \{(x_1, x_5)\}$; $P_{46} = \{(x_3, x_5)\}$ becomes consistent.

References

1. Bana e Costa, C.A., De Corte, J.-M., Vansnick, J.-C.: On the mathematical foundations of MACBETH. In: Figueira, J., Greco, S., Ehrgott, M. (eds.) Multiple Criteria Decision Analysis: State of the Art Surveys, pp. 409–437. Springer, Heidelberg (2005)
2. Chinneck, J.W.: Computer codes for the analysis of infeasible linear programs. Journal of the Operational Research Society 47 (1996)
3. Chinneck, J.W.: An effective polynomial-time heuristic for the minimum cardinality IIS set-covering problem. Annals of Mathematics and Artificial Intelligence 17 (1996)
4. De Corte, J.M.: Un logiciel d'Exploitation d'Information Préférentielles pour l'Aide à la Décision. Bases Mathématiques et Algorithmiques. PhD thesis, University of Mons-Hainaut, Mons (2002)
5. Doignon, J.P.: Thresholds representation of multiple semiorders. SIAM Journal on Algebraic and Discrete Methods 8, 77–84 (1987)
6. Figueira, J.R., Greco, S., Slowinski, R.: Building a set of additive value functions representing a reference preorder and intensities of preference: Grip method. European Journal of Operational Research 195(2), 460–486 (2009)
7. Mayag, B., Grabisch, M., Labreuche, C.: A characterization of the 2-additive Choquet integral through cardinal information. In: Proceedings of EUROFUSE2009 Workshop Preference Modelling and Decision Analysis (CD), Pamplona, Spain, September 16-18 (2009)
8. Mousseau, V., Dias, L., Figueira, J.R.: Dealing with inconsistent judgments in multiple criteria sorting models. 4OR, 4 (2006)
9. Mousseau, V., Figueira, J.R., Dias, L., Da Silva, C.G., Climaco, J.: Resolving inconsistencies among constraints on the parameters of an MCDA model. European Journal of Operational Research 147 (2003)
10. Mousset, C.: Familles de structures de préférence non complètes. PhD thesis, University of Mons-Hainaut, Belgium (2004)
11. Mousset, C.: Families of relations modelling preferences under incomplete information. European journal of operational research 192(2), 538–548 (2009)
12. Mousset, C., Vansnick, J.-C.: About the representation of a precardinal information. Documents d'Economie et de Gestion de l'Université de Mons-Hainaut (2005)
13. Pirlot, M., Vincke, P.: Semiorders — Properties, Representations, Applications. Kluwer Academic Publishers, Dordrecht (1997)